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A string field theory based on causal dynamical triangulations

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Abstract: We formulate the string field theory in zero-dimensional target space corresponding to the two-dimensional quantum gravity theory defined through Causal Dynamical Triangulations. This third quantization of the quantum gravity theory allows us in principle to calculate the transition amplitudes of processes in which the topology of space changes in time, and to include non-trivial topologies of space-time. We formulate the corresponding Dyson-Schwinger equations and illustrate how they can be solved iteratively.

Keywords: Lattice Models of Gravity, 2D Gravity, Models of Quantum Gravity, Statistical Methods.

Contents

1. Introduction

The formulation of a non-critical string theory in which the conformal mode plays an important role dates back to Polyakov. He emphasized the worldsheet formulation of string theory as a two-dimensional quantum gravity theory coupled to matter [1]. This triggered non-perturbative definitions of non-critical string theory [2 – 4], introducing what is now called dynamical triangulation (DT) as a regularization of the worldsheet theory. When the dimension of space-time was larger than 1 these attempts in some sense did not work. One could show that the outcome was not a proper string theory, but a theory where the worldsheet had degenerated into branched polymers [5]. However, when considering matter fields with central charge $c \leq 1$, these regularized theories led to what is now known as non-critical string theory, a very useful toy model of real string theory. In particular, it has been possible to formulate a string field theory of non-critical string theory [6, 7] which is very much simpler than the critical string field theory.

The use of causal dynamical triangulations (CDT) rather than DT as a regularization of quantum gravity was inspired by earlier ideas in [8]: one insists, starting from a Lorentzian space-time, that only causal histories contribute to the quantum gravitational path integral. In addition, one assumes the presence of a global time-foliation. In this way the space-times appearing in the *regularized* path integral become a set of piecewise linear causal geometries, made out of triangles (two-simplices) whose edge lengths provide an ultraviolet cut-off. For a detailed description of how to construct these geometries we refer to [9, 10] in two dimensions and [11] in higher dimensions. In order to perform the summation over these causal geometries we perform a rotation to Euclidean space-times. Each piecewise linear causal geometry as defined in [11] has a continuation to Euclidean signature, but the class of Euclidean geometries included in the path integral will only be a subclass of the total class of Euclidean geometries, and the result of the summation will therefore be different from that of Euclidean quantum gravity.

One is interested in the limit where the lattice spacing a goes to zero. There is evidence for the existence of an underlying (non-perturbatively defined) continuum quantum field theory in four dimensions [12] and the results seem to be in qualitative agreement with recent renormalization group calculations [13]. These intriguing developments in the *four*-dimensional theory are based on numerical simulations, since analytical tools are presently unavailable. In two dimensions the situation is different, since the quantum gravity model can be solved analytically at the discretized level and the limit $a \to 0$ can be constructed.

In [14] we showed that the original two-dimensional CDT model of quantum gravity defined and solved in [9] can be generalized to a model where one allows for the creation of so-called baby universes, branching off from the "parent universe". The creation of a baby universe results in at least one point where from a Lorentzian point of view the metric is degenerate [15]. One cannot invoke the classical theory to decide a priori whether or not such geometries should be included in the path integral. In [9] we made the choice to suppress these configurations. We could also show that if they were completely unsuppressed one would recover Euclidean 2d quantum gravity as defined via DT or quantum Liouville theory. The converse was demonstrated in [16]: if one integrates out all baby universes in Euclidean quantum gravity, one obtains CDT.

Quite surprisingly, there exists yet a third possibility, namely, a double-scaling limit where the creation of baby universes in CDT can be associated with the gravitational coupling constant [14]. In this double-scaling limit one can calculate the disc amplitude and finds a result which is analytically connected to the old CDT result, the expansion parameter being the gravitational coupling constant. However, this cannot — at least not by simple analytic continuation — be connected to the Euclidean theory. Thus we have arrived at a theory which allows the creation of baby universes, but in a much more controlled way than in Euclidean quantum gravity. Of course, unlike the original CDT prescription, this construction contains causality-violating features at the level of the piecewise linear Lorentzian geometries. However, as we will see, the Lorentzian structure still plays a role in "taming" them. — Apart from the interesting observation that such a new theory exists, it may have important implications for the higher-dimensional theories. The attempt to formulate *Euclidean* higher-dimensional quantum gravity theories using DT as a regularization ran into the problem that baby universes completely dominate the path integral and make it difficult to obtain a physically sensible continuum limit. Now we see that there may exist a way to include the creation of baby universes in a controlled manner, starting with the CDT regularization of the quantum gravity theory.

In this paper we show that the construction of [14] can be turned into a full-fledged third quantization of 2d quantum gravity. In the terminology of [6] this is a string field theory for $c = 0$, in the sense that it allows the calculation of amplitudes for splitting and joining of (spatial) universes and as well as the inclusion of different space-time topologies.

The remainder of this article is organized as follows: In section 2 we review briefly the results of the generalized CDT model. In section 3 we show how to define a string field theory, and in section 4 we show how it reproduces the results of the generalized CDT model. In section 5 we derive the general Dyson-Schwinger equations and in section 6 we show how they can be used to calculate in a systematic way multi-universe and topology-changing amplitudes. Finally, we discuss the interpretation and possible generalizations in section 7.

2. Generalized causal dynamical triangulation in 2d

We will initially assume that the two-dimensional space-time has topology $S^1 \times [0,1]$. After rotation to Euclidean signature, the pure gravity action is given by

$$S[g_{\mu\nu}] = \lambda \int \mathrm{d}^2\xi \sqrt{\det g_{\mu\nu}(\xi)} + x \oint \mathrm{d}l_1 + y \oint \mathrm{d}l_2, \tag{2.1}$$

where λ is the cosmological constant, x and y are two so-called boundary cosmological constants, $g_{\mu\nu}$ is the metric of a geometry of the kind described above, and the line integrals refer to the lengths of the in- and out-boundaries induced by $g_{\mu\nu}$. The so-called *proper-time propagator* is defined by

$$G_\lambda(x,y;t) = \int \mathcal{D}[g_{\mu\nu}]\, e^{-S[g_{\mu\nu}]}. \tag{2.2}$$

This represents the Euclideanization of a functional integral over space-times with *Lorentzian* signature, performed over all causal geometries $[g_{\mu\nu}]$ such that the final (or "exit") boundary with boundary cosmological constant y is separated[1] a geodesic distance t from the initial (or "entrance") boundary with boundary cosmological constant x. To arrive at the integral (2.2), all causal geometries have been rotated to Euclidean signature, a procedure which is well defined in the CDT regularization of the path integral.

Calculating the path integral (2.2) with the help of the CDT regularization and taking the continuum limit as the edge length a of the triangles goes to zero leads to the equation

$$\frac{\partial}{\partial t} G_\lambda(x,y;t) = -\frac{\partial}{\partial x}\Big[(x^2-\lambda) G_\lambda(x,y;t)\Big], \tag{2.3}$$

which can readily be solved [9]. Let l_1 denote the length of the initial and l_2 the length of the final boundary. Rather than considering a situation where the boundary cosmological constant x is fixed, we will take l_1 as fixed, and denote the corresponding propagator by

[1] The statement that the exit boundary is separated by a geodesic distance t from the entrance boundary means in this context that *all* points on the exit boundary have a geodesic distance t to the entrance boundary. The geodesic distance of a point on the exit loop to the entrance loop is defined as the minimal geodesic distance from the exit point to points on the entrance loop. In the piecewise flat, triangulated geometries we are working with, "distance" is given by "link distance". In the case of the original, "pure" CDT without any causality violations, the notion of distance between boundaries just introduced is *symmetric* under exchange of entrance and exit boundary. For the generalized models discussed below, this will no longer be the case.

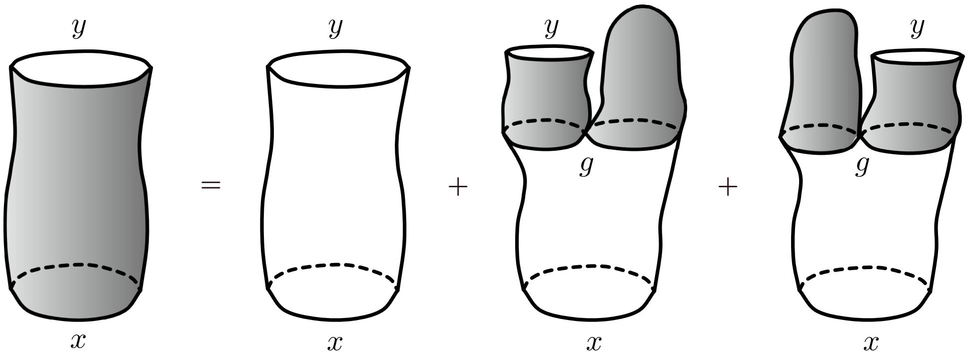

Figure 1: In all four graphs, the geodesic distance from the final to the initial loop is given by t. Differentiating with respect to t leads to eq. (2.6). Shaded parts of graphs represent the full, g-dependent propagator $G_{\lambda,g}$ and disc amplitude $W_{\lambda,g}$, and non-shaded parts the CDT propagator G_λ.

$G_\lambda(l_1, y; t)$, with similar definitions for $G_\lambda(x, l_2; t)$ and $G_\lambda(l_1, l_2; t)$. All of them are related by Laplace transformations, for instance,

$$G_\lambda(x, y; t) = \int_0^\infty \mathrm{d}l_2 \int_0^\infty \mathrm{d}l_1 \; G_\lambda(l_1, l_2; t) \; \mathrm{e}^{-xl_1 - yl_2}. \tag{2.4}$$

Next, we will turn our attention to the so-called *disc amplitude*, associated with a piece of space-time which has the topology of a disc. Strictly speaking, the disc amplitude does not exist in CDT. A spatial slice in a two-dimensional Lorentzian space-time of the type we are considering will by construction be a one-dimensional space-like subspace of topology S^1, i.e. a circle. Now, there is no way this can be extended to a well-defined Lorentzian geometry everywhere in the interior of any finite disc whose boundary is the circle. The light cones of the geometry must degenerate in at least a point, because the disc does not extend infinitely in time. However, after rotation to Euclidean signature,[2] we can *define* a disc amplitude, which is related to $G_\lambda(x, l_2; t)$ by

$$W_\lambda(x) = \int_0^\infty dt \; G_\lambda(x, l_2 = 0; t) = \frac{1}{x + \sqrt{\lambda}}. \tag{2.5}$$

There is clearly a latest time t where the spatial universe contracts to length zero and vanishes into the "vacuum". When introducing the string field theory below, we will see that this process has a natural realization as a tadpole term in the string field Hamiltonian.

We will now allow for the possibility that space branches into disconnected parts as a function of proper time t, and introduce a coupling constant g of mass dimension

[2]We should emphasize that there is in principle a choice involved when generalizing the unique Wick rotation of CDT [9 – 11] to situations where the causal structure of the piecewise flat geometries has singularities. One might attach certain complex and/or singular weights to such singularities in the Euclideanization, for example, of the kind envisaged in [15]. When the disc amplitude was first introduced in two-dimensional CDT in order to compare it to Euclidean models [9], no extra weight was associated with it, leading to the disc amplitude (2.5). In the present work, following [17, 14], we will associate finite, real weights with baby universes and branching points, as will be explained in detail below.

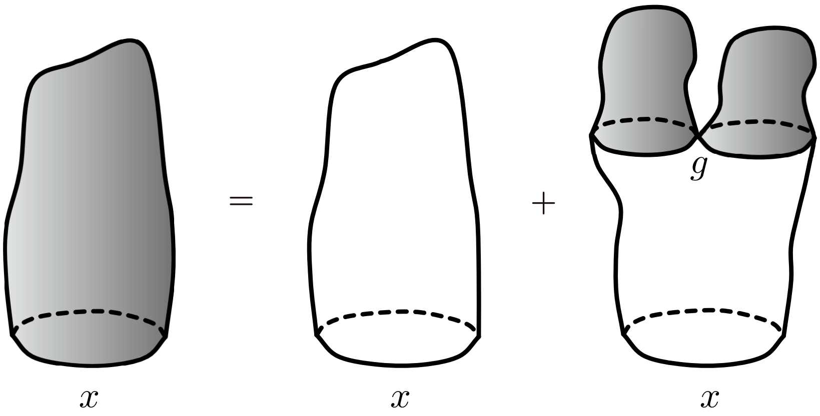

Figure 2: Graphical illustration of eq. (2.7). Shaded parts represent the full disc amplitude $W_{\lambda,g}$, unshaded parts the CDT disc amplitude W_λ and propagator G_λ.

3 associated with the branching.[3] As shown in [14], this modifies the equation for the proper-time propagator to

$$\frac{\partial}{\partial t} G_{\lambda,g}(x,y;t) = -\frac{\partial}{\partial x}\Big[\Big((x^2-\lambda) + 2g\, W_{\lambda,g}(x)\Big) G_{\lambda,g}(x,y;t)\Big], \tag{2.6}$$

where the generalized nature of the propagator $G_{\lambda,g}$ is indicated by the additional subscript g. The graphical representation of the integral version of eq. (2.6) is shown in figure 1. At this point, the new, generalized disc amplitude $W_{\lambda,g}(x)$ is unknown and has to satisfy the equation

$$W_{\lambda,g}(x) = W^{(0)}_{\lambda,g}(x) + g \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}l_1 \mathrm{d}l_2\; (l_1+l_2) G^{(0)}_{\lambda,g}(x, l_1+l_2; t) W_{\lambda,g}(l_1) W_{\lambda,g}(l_2), \tag{2.7}$$

where superscripts (0) indicate the CDT amplitudes introduced in eqs. (2.5) and (2.3) above, that is,

$$W^{(0)}_{\lambda,g}(x) \equiv W_{\lambda,g=0}(x) = W_\lambda(x), \tag{2.8}$$

and similarly for $G^{(0)}_{\lambda,g}$. The graphical representation of eq. (2.7) is shown in figure 2. The integrations in (2.7) can be performed and one finds [14]

$$\hat{W}_{\lambda,g}(x) = (x-c)\sqrt{(x+c)^2 - \frac{2g}{c}}, \quad c = u\sqrt{\lambda}, \quad u^3 - u + \frac{g}{\lambda^{3/2}} = 0, \tag{2.9}$$

where

$$\hat{W}_{\lambda,g}(x) \equiv (x^2-\lambda) + 2g W_{\lambda,g}(x). \tag{2.10}$$

[3] One could in principle have considered a more general branching process, where more than one baby universe can sprout at any given time step t. However, starting with the discretized theory and a lattice cut off, one can show that such processes are suppressed when the lattice spacing goes to zero [14]. This is related to the fact that g has mass dimension 3.

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Using the definition (2.10) and eq. (2.9), we can write eq. (2.6) as

$$\frac{\partial}{\partial t} G_{\lambda,g}(x,y;t) = -\frac{\partial}{\partial x}\Big[\hat{W}_{\lambda,g}(x)\, G_{\lambda,g}(x,y;t)\Big]. \tag{2.11}$$

The solution is

$$G_{\lambda,g}(x,y;t) = \frac{\hat{W}_{\lambda,g}(\bar{x}(t,x))}{\hat{W}_{\lambda,g}(x)}\,\frac{1}{\bar{x}(t,x)+y}, \tag{2.12}$$

where $\bar{x}(t,x)$ is the solution of the characteristic equation for (2.11),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = -\hat{W}_{\lambda,g}(\bar{x}), \quad \bar{x}(0,x) = x. \tag{2.13}$$

The generalized CDT model of 2d quantum gravity we have defined above is a perturbative deformation of the original model in the sense that it has a convergent power expansion of the form

$$W_{\lambda,g}(x) = \frac{1}{\sqrt{\lambda}}\sum_{n=0}^{\infty} c_n\left(\frac{x}{\sqrt{\lambda}}\right)\tilde{g}^n, \quad \tilde{g} \equiv \left(\frac{g}{\lambda^{3/2}}\right), \tag{2.14}$$

in the dimensionless coupling constant $g/\lambda^{3/2}$. This implies in particular that the average number $\langle n\rangle$ of baby universes created during the proper-time evolution of the two-dimensional universe described by this model is finite, a property already observed in previous 2d models with topology change [17]. The expectation value of the number n of branchings can be computed according to

$$\langle n\rangle = \frac{g}{W_{\lambda,g}(x)}\frac{\mathrm{d}W_{\lambda,g}(x)}{\mathrm{d}g}, \tag{2.15}$$

which is finite as long as we are in the range of convergence of $W_{\lambda,g}(x)$. This coincides precisely with the range where the function $W_{\lambda,g}(x)$ behaves in a physically acceptable way, namely, $W_{\lambda,g}(l)$ goes to zero if the length l of the boundary loop goes to infinity [14].

Is it possible to give a gravitational interpretation of the new coupling constant g? From a purely Euclidean point of view all graphs appearing in figure 2 have the fixed topology of a disc. However, from a Lorentzian point of view, which comes with a notion of time, it is clear that the branching of a baby universe is associated with a change of the *spatial* topology, a singular process in a Lorentzian space-time [15]. One way of keeping track of this in a Wick-rotated, Euclidean picture is as follows. Since each time a baby universe branches off it also has to end somewhere, we may think of marking the resulting "tip" with a puncture. (Of course, these baby universes can in turn have baby universes branching off them, giving rise to additional branchings and punctures.) From a gravitational viewpoint, each new puncture corresponds to a topology change and receives a weight $1/G_N$, where G_N is Newton's constant, because it will lead to a change by precisely this amount in the two-dimensional (Euclidean) Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\pi G_N}\int \mathrm{d}^2\xi\sqrt{g}R. \tag{2.16}$$

Let us view the continuum theory as the limit of a lattice theory (CDT) with lattice spacing a. On the lattice we have a dimensionless "bare" coupling constant $g_0(a) = ga^3$, where a is the lattice spacing (see [14] for a detailed discussion). According to the arguments above we can now make the identification $g_0(a) = e^{-1/G_N(a)}$, where $G_N(a)$ denotes the "bare" gravitational coupling constant. In the regularized lattice version the "bare" dimensionless cosmological coupling constant λ_0 will approach a critical value λ_c and the continuum, dimensionful coupling constant λ which appears in this limit is defined by $\lambda_0 - \lambda_c = \lambda a^2 + O(a^3)$ (see [9] for details). In fact this relation defines how the lattice spacing should be taken to zero when λ_0 goes to λ_c in order that a continuum, dimensionful λ survives in the continuum, or vice versa it defines the behavior of the bare coupling constant λ_0 as a function $\lambda_0(a)$ of the cut off a. Returning to the expression for $W_{\lambda,g}(x)$ in (2.14), the expansion parameter is, as already noticed, not really g but the dimensionless $g/\lambda^{3/2}$. In the regularized lattice theory the following combination appears (see again [14] for a detailed discussion)

$$\frac{g_0(a)}{(\lambda_0(a) - \lambda_c)^{3/2}} \to \frac{g}{\lambda^{3/2}} \tag{2.17}$$

for $a \to 0$. We can thus write

$$\frac{1}{G_N(a)} + \ln(\lambda_0(a) - \lambda_c)^{3/2} \to -\ln\left(g/\lambda^{3/2}\right), \tag{2.18}$$

for $a \to 0$ This makes it natural to introduce a *renormalized* gravitational coupling constant by

$$\frac{1}{G_N^{\mathrm{ren}}} = \frac{1}{G_N(a)} + \ln(\lambda_0(a) - \lambda_c)^{3/2} = \frac{1}{G_N(a)} + \frac{3}{2}\ln \lambda a^2. \tag{2.19}$$

This implies that the *bare* gravitational coupling constant $G_N(a)$ goes to zero like $1/|\ln a^3|$ when the cut-off vanishes, $a \to 0$, in such a way that the product $\mathrm{e}^{1/G_N^{\mathrm{ren}}}/\lambda^{3/2}$ is independent of the cut-off a. We can now identify

$$\mathrm{e}^{-1/G_N^{\mathrm{ren}}} = g/\lambda^{3/2} \tag{2.20}$$

as the genuine coupling parameter in which we expand.

This renormalization of the gravitational (or string) coupling constant is reminiscent of the famous double-scaling limit in non-critical string theory [18]. In that case one also has $g \propto e^{-1/G_N^{\mathrm{ren}}}$, the only difference being that relation (2.19) is changed to

$$\frac{1}{G_N^{\mathrm{ren}}} = \frac{1}{G_N(a)} + \frac{5}{4}\ln \lambda a^2, \tag{2.21}$$

whence the partition function of non-critical string theory appears precisely as a function of the dimensionless coupling constant $g/\lambda^{5/4}$.

3. String field theory

In quantum field theory, particles can be created and annihilated if the process does not violate any conservation laws of the theory. In string field theory, one has analogous operators which can create and annihilate strings. From the 2d quantum gravity point of view we are dealing with a third quantization of gravity: one-dimensional universes can be created and destroyed. In [6] such a formalism was developed for non-critical strings in a zero-dimensional target space (or 2d Euclidean quantum gravity). We will follow the procedure outlined there closely and develop a string field theory or third quantization for CDT, which will allow us in principle to calculate any amplitude involving the creation or annihilation of universes.

As starting point we assume the existence of a vacuum from which universes can be created. We denote this state $|0\rangle$ and define creation and annihilation operators through

$$[\Psi(l), \Psi^\dagger(l')] = l\delta(l-l'), \quad \Psi(l)|0\rangle = \langle 0|\Psi^\dagger(l) = 0. \tag{3.1}$$

This assignment corresponds to working with spatial universes where a point has been marked. This is merely a formal aspect, to avoid having to put in certain combinatorial factors by hand when gluing universes together. The operators $\Psi(l)$ and $\Psi^\dagger(l)$ will be assigned the dimensions $\dim[\Psi] = \dim[\Psi^\dagger] = 0$.

We could alternatively have chosen creation and annihilation operators which create and annihilate universes without such a mark. Instead of (3.1) we then would have had

$$[\Psi(l), \Psi^\dagger(l')] = l^{-1}\delta(l-l'), \quad \Psi(l)|0\rangle = \langle 0|\Psi^\dagger(l) = 0, \tag{3.2}$$

with corresponding dimensional assignments $\dim[\Psi] = 1$ and $\dim[\Psi^\dagger] = 1$. One could even let $\Psi^\dagger$ create marked universes and Ψ annihilate unmarked universes if one compensated for the missing combinatorial factors by hand. In the following we will use the assignment (3.1).

Let us write the propagator equation (2.3) using the boundary length rather than the boundary cosmological constant as a variable,[4] that is,

$$\frac{\partial}{\partial t}\tilde{G}_\lambda(l_1, l_2; t) = l_1\left(\frac{\partial^2}{\partial l_1^2} - \lambda\right)\tilde{G}_\lambda(l_1, l_2; t), \tag{3.3}$$

which we can also write as

$$\tilde{G}_\lambda(l_1, l_2; t) = \langle l_2|\mathrm{e}^{-tH_0(l)}|l_1\rangle, \qquad H_0(l) = -l\frac{\partial^2}{\partial l^2} + \lambda l. \tag{3.4}$$

Associated with the spatial universe we have a Hilbert space on the positive half-line, and a corresponding scalar product

$$\langle\psi_1|\psi_2\rangle = \int \frac{dl}{l}\, \psi_1^*(l)\psi_2(l), \tag{3.5}$$

which makes $H_0(l)$ hermitian. The introduction of the operators $\Psi(l)$ and $\Psi^\dagger(l)$ in (3.1) can be thought of as analogous to the standard second quantization in many-body theory. The single-particle Hamiltonian becomes in our case the "single-universe" Hamiltonian

[4]For convenience of notation we have in (3.3) also marked the exit loop l_2 in order to have symmetry between the loops at initial and final time, i.e. $\tilde{G}_\lambda(l_1, l_2; t) = l_2 G_\lambda(l_1, l_2; t)$.

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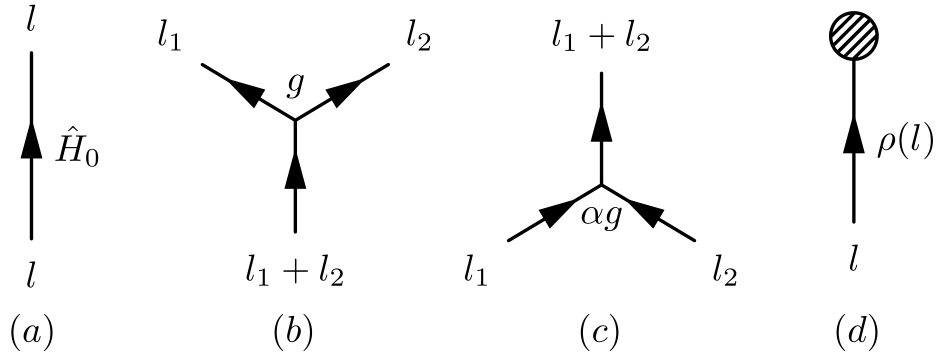

Figure 3: The elementary terms of the string field theory Hamiltonian (3.10): (a) the single spatial universe propagator, (b) the term corresponding to splitting into two spatial universes, (c) the term corresponding to the merging of two spatial universes and (d) the tadpole term.

$H_0(l)$. It has normalized eigenfunctions $\psi_n(l)$ with corresponding eigenvalues $e_n = 2n\sqrt{\lambda}$, $n = 1, 2, \ldots$,

$$\psi_n(l) = l\, e^{-\sqrt{\lambda} l} p_{n-1}(l), \qquad H_0(l)\psi_n(l) = e_n \psi_n(l), \tag{3.6}$$

where $p_{n-1}(l)$ is a polynomial of order $n-1$. We now introduce creation and annihilation operators $a_n^\dagger$ and a_n corresponding to these states, acting on the Fock vacuum $|0\rangle$ and satisfying $[a_n, a_m^\dagger] = \delta_{nm}$. We define

$$\Psi(l) = \sum_n a_n \psi_n(l), \qquad \Psi^\dagger(l) = \sum_n a_n^\dagger \psi_n^*(l), \tag{3.7}$$

and from the orthonormality of the eigenfunctions with respect to the measure dl/l we recover (3.1). The "second-quantized" Hamiltonian is given by

$$\hat{H}_0 = \int_0^\infty \frac{dl}{l}\, \Psi^\dagger(l) H_0(l) \Psi(l), \tag{3.8}$$

and the propagator $\tilde{G}_\lambda(l_1, l_2; t)$ is now obtained as

$$\tilde{G}_\lambda(l_1, l_2; t) = \langle 0|\Psi(l_2) \mathrm{e}^{-t\hat{H}_0} \Psi^\dagger(l_1)|0\rangle. \tag{3.9}$$

While all of this is rather straightforward, the advantage of the formalism is that it automatically takes care of symmetry factors (like in the many-body applications in statistical field theory), both when many spatial universes are at play and when they are joining and splitting. Following [6], we define the Hamiltonian

$$\begin{aligned} \hat{H} \;=\; & \hat{H}_0 + g \int dl_1 \int dl_2 \Psi^\dagger(l_1)\Psi^\dagger(l_2)\Psi(l_1 + l_2) \\ & -\alpha g \int dl_1 \int dl_2 \Psi^\dagger(l_1 + l_2)\Psi(l_2)\Psi(l_1) - \int \frac{dl}{l}\, \rho(l)\Psi(l), \end{aligned} \tag{3.10}$$

describing the interaction between spatial universes (the different terms are illustrated in figure 3). Here g is the coupling constant of mass dimension 3 we have already encountered in section 2, and $\rho(l)$ denotes the amplitude for a universe component of length l to disappear into the vacuum. The factor α has merely been introduced to distinguish between the action of the two terms proportional to g in (3.10) when expanding in powers of g. We will usually assume $\alpha = 1$, unless explicitly stated otherwise. Note that the signs of all the interaction terms in (3.10) are negative. This reflects the fact that we want these terms to represent the insertion of new geometric structures compared to the "free" propagation generated by $\hat{H}_0$. These structures should therefore appear with positive weight when we expand $e^{-t\hat{H}}$.

The Hamiltonian $\hat{H}$ is hermitian except for the presence of the tadpole term. It tells us that universes can vanish, but not be created from nothing. The meaning of the two interaction terms is as follows. The first term replaces a universe of length $l_1 + l_2$ with two universes of length l_1 and l_2. This is precisely the process shown in figure 1. The second term represents the opposite process where two spatial universes merge into one, i.e. the time-reversed picture. The coupling constant g is seen to appear as a string coupling constant: one factor of g for the splitting, and another factor of g for the merging of spatial universes, leading to a combined factor g^2 whenever a handle is added to the space-time.

In a way the appearance of a tadpole term is more natural in CDT than in the original Euclidean framework in [6], since in CDT it has its origin in a physical, causality-violating process located at the end point (in time) of the disc, where the baby universe disappears into nothing, as we saw in section 2. The tadpole term is a formal realization of this. Because of the one-to-one correspondence between punctures and baby universe branchings, we can also associate this process with the gravitational coupling constant, in this way linking it to g. The shift in associating the coupling g from the splitting of spatial universes to the vanishing of universes can be made explicit in our string field Hamiltonian $\hat{H}$ in (3.10). In (3.10), the coupling constant g was associated with the splitting and joining of spatial universes, but no coupling constant with the tadpole term, i.e. the vanishing of a spatial universe. However, by redefining Ψ and $\Psi^\dagger$ to

$$\bar{\Psi} = \frac{1}{g}\,\Psi, \qquad \bar{\Psi}^\dagger = g\,\Psi^\dagger, \tag{3.11}$$

the coupling constant g is shifted from the splitting to the tadpole term, i.e. precisely the shift mentioned above. In addition, the term associated with the joining of spatial universes will have the coupling constant g^2, which precisely accounts for the change in topology.

Finally, let us identify the true, dimensionless coupling constant governing (3.10). This can be done by re-expressing everything in units of $1/\sqrt{\lambda}$, which represents the natural length scale of our universe. Introducing the dimensionless length variable $\tilde{l} = l\sqrt{\lambda}$, the dimensionless boundary cosmological constant $\tilde{x} = x/\sqrt{\lambda}$, the dimensionless time variable $\tilde{t} = t\sqrt{\lambda}$, the dimensionless tadpole density $\tilde{\rho}(\tilde{l}) = \rho(l)/\sqrt{\lambda}$, the dimensionless coupling constant $\tilde{g} = g/\lambda^{3/2}$ (already introduced in eq. (2.14)), and finally the dimensionless Hamiltonian $\tilde{H} = \hat{H}/\sqrt{\lambda}$, we can write

$$\hat{H}_0 = \sqrt{\lambda}\,\tilde{H}_0, \quad \tilde{H}_0 = \int \frac{d\tilde{l}}{\tilde{l}}\tilde{\Psi}^\dagger(\tilde{l})H_0(\tilde{l})\tilde{\Psi}(\tilde{l}), \tag{3.12}$$

where $\tilde{\Psi}(\tilde{l}) = \Psi(l)$ and $\tilde{\Psi}^\dagger(\tilde{l}) = \Psi^\dagger(l)$ satisfy the same commutation relation as $\Psi(l), \Psi^\dagger(l)$ when expressed in terms of $\tilde{l}$, and $\hat{H} = \sqrt{\lambda}\tilde{H}$, where

$$\begin{aligned}\tilde{H} &= \tilde{H}_0 - \tilde{g} \int d\tilde{l}_1 \int d\tilde{l}_2 \tilde{\Psi}^\dagger(\tilde{l}_1)\tilde{\Psi}^\dagger(\tilde{l}_2)\tilde{\Psi}(\tilde{l}_1+\tilde{l}_2) \\ &\quad -\alpha\tilde{g} \int d\tilde{l}_1 \int d\tilde{l}_2 \tilde{\Psi}^\dagger(\tilde{l}_1+\tilde{l}_2)\tilde{\Psi}(\tilde{l}_2)\tilde{\Psi}(\tilde{l}_1) - \int \frac{d\tilde{l}}{\tilde{l}}\, \tilde{\rho}(\tilde{l})\tilde{\Psi}(\tilde{l}).\end{aligned} \tag{3.13}$$

From this expression we can read off that the true coupling constant of the theory is the dimensionless quantity $\tilde{g}$, precisely the "double-scaling" coupling constant which already appeared in the calculation of $W_{\lambda,g}(x)$ and $G_{\lambda,g}(x,y)$, c.f. eq. (2.14). From the discussion above we also observe that the parameter associated with a topological expansion of space-time is given by $\tilde{g}^2 = g^2/\lambda^3$. In principle we can now calculate any process which starts from m spatial universes at time 0 and ends with n universes at time t, represented by the amplitude

$$\tilde{G}_{\lambda,g}(l_1,..,l_m;l'_1,..,l'_n;t) = \langle 0|\Psi(l'_1)\ldots\Psi(l'_n)\, e^{-t\hat{H}}\Psi^\dagger(l_1)\ldots\Psi^\dagger(l_m)|0\rangle. \tag{3.14}$$

4. The $\alpha = 0$ limit

4.1 The disc amplitude

Let us now consider the simplest such amplitude, that of a single spatial universe disappearing into the vacuum. This is precisely the disc amplitude of generalized CDT considered in section 2. There, we allowed baby universes to branch off, but they were forbidden to rejoin the parent universe, and thus were destined to disappear into the vacuum eventually. In other words, the topology of *space-time* was not allowed to change during evolution. This can be incorporated in the string field-theoretic picture by choosing $\alpha = 0$ in (3.10). The disc amplitude can then be expressed as

$$W_{\lambda,g}(l) = \lim_{t\to\infty} W_{\lambda,g}(l,t) = \lim_{t\to\infty} \langle 0|\,\mathrm{e}^{-t\hat{H}(\alpha=0)}\Psi^\dagger(l)|0\rangle. \tag{4.1}$$

It describes all possible ways in which a spatial loop can develop in time and disappear into the vacuum without changing the topology of space-time. Note that the tadpole term in (3.10) is needed if the amplitude (4.1) should be different from zero, since the state $|l\rangle = \Psi^\dagger(l)|0\rangle$ is orthogonal to the vacuum state $|0\rangle$. We note that for $\alpha = 0$ the vacuum expectation value

$$\begin{aligned}&\langle 0|\,\mathrm{e}^{-t\hat{H}(\alpha=0)}\Psi^\dagger(l_1)\cdots\Psi^\dagger(l_m)|0\rangle = \\ &\quad\langle 0|\,\mathrm{e}^{-t\hat{H}(\alpha=0)}\Psi^\dagger(l_1)|0\rangle\langle 0|\,\mathrm{e}^{-t\hat{H}(\alpha=0)}\Psi^\dagger(l_2)|0\rangle\langle 0|\cdots|0\rangle\langle 0|\,\mathrm{e}^{-t\hat{H}(\alpha=0)}\Psi^\dagger(l_m)|0\rangle\end{aligned} \tag{4.2}$$

factorizes, as one can easily prove using the algebra of the Ψ's. This is an expression of the fact that if we start out with m spatial universes, there is no way they can merge at any time if $\alpha = 0$.

Following [6], we obtain an equation for $W_{\lambda,g}(l)$ by differentiating (4.1) with respect to t and using that $\hat{H}|0\rangle = 0$,

$$0 = \lim_{t\to\infty} \frac{\partial}{\partial t} W_{\lambda,g}(l,t) = \lim_{t\to\infty} \langle 0|e^{-t\hat{H}(\alpha=0)}[\hat{H}(\alpha=0),\Psi^\dagger(l)]|0\rangle. \tag{4.3}$$

The commutator can readily be calculated and after a Laplace transformation eq. (4.3) reads

$$\frac{\partial}{\partial x}\left((x^2-\lambda)W_{\lambda,g}(x)+gW_{\lambda,g}^2(x)\right)=\rho(x), \tag{4.4}$$

where the last term on the left-hand side of eq. (4.4) is a consequence of the factorization (4.2). Eq. (4.4) has the generalized CDT solution (2.9)-(2.10) discussed in section 2 if

$$\rho(x)=1, \quad \text{i.e.} \quad \rho(l)=\delta(l). \tag{4.5}$$

This is a reasonable physical requirement, which we will implement in what follows: the spatial universe can only vanish into the vacuum when the length of the universe goes to zero.

4.2 Inclusive amplitudes

After reproducing the generalized CDT disc amplitude $W_{\lambda,g}(x)$ as the connected amplitude arising in the string field theory in the limit $\alpha=0$, we now want to understand how to rederive the proper-time propagator $\tilde{G}_{\lambda,g}(x,y,t)$ in this context. This propagator is characterized by an entrance loop at time $t=0$ and an exit loop at time t, and also contains baby universes which branch off and can extend in time *beyond* time t, if only they vanish into the vacuum eventually, as indicated in figure 1.

We can reproduce this result in the $\alpha=0$ limit of the string field theory by introducing the so-called "inclusive" Hamiltonian [6]. Since we are working in the $\alpha=0$ limit, universes can only branch and not merge during the time evolution, and all but one have to vanish into the vacuum. The branching process is associated with the term

$$g\int dl_1\int dl_2\Psi^\dagger(l_1)\Psi^\dagger(l_2)\Psi(l_1+l_2) \tag{4.6}$$

in the Hamiltonian $\hat{H}$ of eq. (3.10). Once the branching has occurred, only one of the two universes can connect to the exit loop at time t, the other one has to continue until it eventually vanishes into the vacuum, a process which may occur at a time later than t. This scenario is captured by replacing

$$\Psi^\dagger(l_1)\Psi^\dagger(l_2)\to W_{\lambda,g}(l_1)\Psi^\dagger(l_2)+\Psi^\dagger(l_1)W_{\lambda,g}(l_2) \tag{4.7}$$

in eq. (4.6), thus arriving at the "inclusive Hamiltonian"

$$\hat{H}_{\mathrm{incl}}=\int\frac{dl}{l}\Psi^\dagger(l)H_0(l)\Psi(l)-2g\int dl_1\int dl_2\, W_{\lambda,g}(l_1)\Psi^\dagger(l_2)\Psi(l_1+l_2), \tag{4.8}$$

which enables us to rewrite the corresponding propagator $\tilde{G}_{\lambda,g}(l_1,l_2;t)$ as

$$\tilde{G}_{\lambda,g}(l_1,l_2;t)=\langle 0|\Psi(l_2)\,\mathrm{e}^{-t\hat{H}_{\mathrm{incl}}}\Psi^\dagger(l_1)|0\rangle. \tag{4.9}$$

Differentiating eq. (4.9) with respect to t, commuting $\hat{H}_{\mathrm{incl}}$ through to the right and using $\hat{H}_{\mathrm{incl}}|0\rangle=0$, one obtains after a Laplace transformation eq. (2.6). We conclude that also the generalized CDT proper-time propagator has a simple string field-theoretic description.

4.3 Propagator identities

Our starting point was the functional integral (2.2) over all two-dimensional geometries with cylindrical topology, where the entrance and exit loop were separated by a geodesic distance t. This proper-time propagator played an important role, motivating the introduction of the string field Hamiltonian. As explained in footnote 1, this construction is based on a particular definition of the geodesic distance between the exit and the entrance loop: every point on the exit loop has geodesic distance t to the entrance loop, i.e. the minimal distance of a given point on the exit loop to the points on the entrance loop is precisely t, independent of the point on the exit loop. This implies that the exit loop as a whole has a specific, "parallel" orientation relative to the entrance loop. This is a very useful property for the propagator to have, ensuring the existence of simple composition rules and thus a Hamiltonian.

What we will show next is that one can define more general amplitudes, which depend on a somewhat looser notion of distance between their boundary components, and which are obtained by appropriately gluing together proper-time propagators and disc amplitudes. These "combined" propagators obey non-trivial identities analogous to identities first found and verified in the Euclidean framework of the non-critical string field theory of [6]. The fact that our CDT geometries still carry some memory of their original Lorentzian structure after mapping them to the Euclidean sector makes the physical interpretation of these identities in the CDT string field theory less clear, since the nature of the identities is rather "Euclidean", as we shall see.

The geometric configurations we are interested in consist of two entrance loops from which two universes propagate to the future, and then join to form a single universe, which eventually disappears into the vacuum. Two distinct configurations of this type are illustrated in figure 4. They differ in how much time elapses in each of the "legs" before they join. When summing over all geometries of fixed leg lengths (t_1, t_2), the legs will correspond to proper-time propagators of length t_1 and t_2, and the remainder of the geometry will correspond to a disc amplitude with boundary length $l + l'$, which has been pinched in a point such that it can be glued to the two exit loops of the propagators, of length l and l' respectively. We will be interested in comparing situations where the two leg lengths sum to the same number t, such that $t_2 = t - t_1$, for different t_1. The left illustration in figure 4 corresponds to the extreme case $t_1 = 0$, and the right one to some intermediate choice $t_1 < t_2$. We are not primarily concerned with the physicality or otherwise of these geometries, but simply note that they are well defined in our string field-theoretic set-up after Euclideanization, and possess calculable amplitudes. We can allow the propagation to be of the most general $\alpha = 0$ kind,[5] such that the dynamics is described by the inclusive Hamiltonian $\hat{H}_{\mathrm{incl}}$. The two situations depicted in figure 4 correspond to the two calculations

$$\int_0^\infty \mathrm{d}l \; \tilde{G}_{\lambda,g}(l_1, l; t) W_{\lambda,g}(l + l_2) \tag{4.10}$$

[5]Strictly speaking, the processes described by (4.10) and (4.11) are of order α in the string field-theoretic terminology we have introduced above, since they describe two merging spatial universes. A related amplitude $w_0(l_1, l_2)$, obtained by integrating over all times t, will be introduced later, c.f. (6.21).

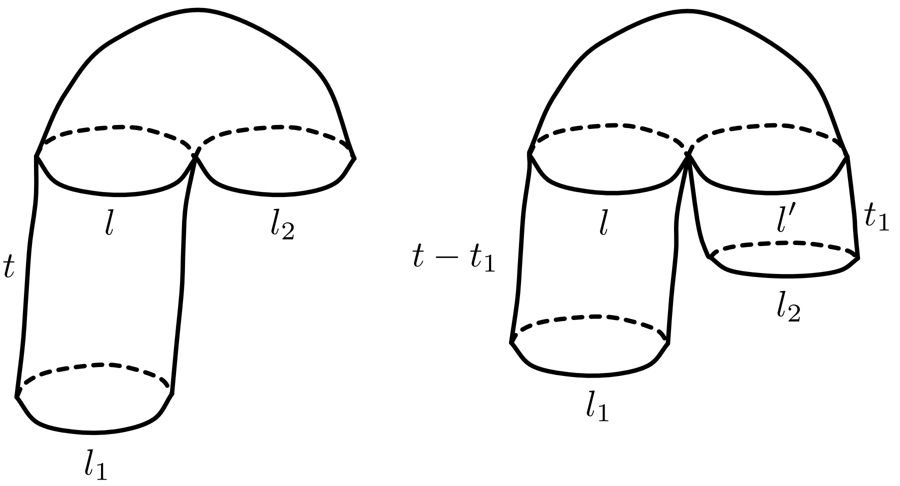

Figure 4: Two universes (whose time extensions add up to t) merging into a single one, and subsequently vanishing into the vacuum. The figure on the left shows the degenerate case where one leg has length t and the other length 0, whereas the figure on the right has two legs of unequal, non-zero length. An explicit computation shows that summing over all space-times of the first type gives the same result as summing over all space-times of the second type, for any choice of t_1.

and

$$\int_0^\infty \mathrm{d}l \int_0^\infty dl' \; \tilde{G}_{\lambda,g}(l_1,l;t_1) W_{\lambda,g}(l+l') \tilde{G}_{\lambda,g}(l_2,l';t-t_1), \tag{4.11}$$

for some $0 < t_1 < t$. The remarkable fact is that the results of both calculations coincide! Equivalently, one can show that

$$0 = \frac{\partial}{\partial t_1} \int_0^\infty dl \int_0^\infty dl' \; \tilde{G}_{\lambda,g}(l_1,l;t_1) W_{\lambda,g}(l+l') \tilde{G}_{\lambda,g}(l_2,l';t-t_1). \tag{4.12}$$

After a Laplace transformation, eq. (4.12) reads

$$0 = \frac{\partial}{\partial t_1} \int_{-i\infty+c}^{i\infty+c} \frac{dz}{2\pi i} \; \tilde{G}_{\lambda,g}(x,-z;t_1) W_{\lambda,g}(z) \tilde{G}_{\lambda,g}(y,-z;t-t_1). \tag{4.13}$$

Using the explicit form of $G_{\lambda,g}(x,y;t)$, eq. (2.12), we can perform the z-integration in eq. (4.13). Next, with the help of eq. (2.10), we can express $W_{\lambda,g}(x)$ in terms of $\hat{W}_{\lambda,g}(x)$ (given by eq. (2.7)), and finally, using eq. (2.13), we can perform the t_1-differentiation. The result is

$$0 = \frac{\partial^2}{\partial x \partial y} \Big(\bar{x}^2(t_1,x) - \bar{y}^2(T-t_1,y) \Big), \tag{4.14}$$

which is satisfied. The upshot of this calculation is that we can define a more general amplitude

$$\mathcal{G}(l_1,l_2,t) := \int_0^\infty dl \int_0^\infty dl' \; \tilde{G}_{\lambda,g}(l_1,l;t_1) W_{\lambda,g}(l+l') \tilde{G}_{\lambda,g}(l_2,l';t-t_1) \tag{4.15}$$

associated with this merger process, which only depends on the combined distance t along the legs, and which — as we have just proved — is invariant under how t is split into two. It is somewhat surprising that this invariance property holds, since as Lorentzian geometries the two situations depicted in figure 4 are clearly distinct. Of course, during the Wick rotation the special character of the causality-violating merger point between the two "trouser legs" disappears, which may explain the validity of (4.10), just like in the Euclidean formulation.[6] In line with the latter, one may interpret the quantity $\mathcal{G}(l_1, l_2, t)$ as a generalized amplitude with two boundaries separated by a distance t, where the "distance" between two spatial loops is now defined as the smallest geodesic distance between *any* pair of points on the two loops, with no further constraints on the relative position of the two loops. In particular, this makes $\mathcal{G}(l_1, l_2, t)$ symmetric under the exchange of l_1 and l_2.

5. Dyson-Schwinger equations

The disc amplitude $W_{\lambda,g}$ is one of a set of functions for which it is possible to derive Dyson-Schwinger equations. Here we will consider a more general class of functions. Defining the generating function $Z(J;t)$ by

$$Z(J;t) = \langle 0|\mathrm{e}^{-t\hat{H}}\ \mathrm{e}^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle, \tag{5.1}$$

we have

$$\langle 0|\mathrm{e}^{-t\hat{H}}\ \Psi^\dagger(l_1)\cdots\Psi^\dagger(l_n)|0\rangle = \left.\frac{\delta^n Z(J;t)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0}. \tag{5.2}$$

For the special case of vanishing coupling $\alpha = 0$, we have already seen that the amplitudes factorize, such that

$$Z(J,t;\alpha=0) = \mathrm{e}^{\int dl\, J(l)W_{\lambda,g}(l,t)}, \tag{5.3}$$

where $W_{\lambda,g}(l,t)$ denotes the disc amplitude where the universe decays into the vacuum before or at time t, and where $W_{\lambda,g}(l, t = \infty)$ is the disc amplitude we have already calculated.

Following [6], we can obtain the Dyson-Schwinger equations in the same way as for the disc amplitude, the only difference being that when the constant α is no longer zero, these equations do not close but connect various amplitudes of more complicated topology. However, as we shall see, the equations can still be solved iteratively. We denote

$$Z(J) \equiv \lim_{t\to\infty} Z(J;t), \tag{5.4}$$

[6] The corresponding equation in the case of non-critical string theory is

$$0 = \frac{\partial^2}{\partial x \partial y}\Big(\bar{x}(t_1, x) - \bar{y}(t - t_1, y)\Big),$$

leading again to the result that the amplitude $\mathcal{G}(l_1, l_2; t)$ is independent of the subdivision of $t = t_1 + t_2$. Note that in this purely Euclidean formulation a relation like (4.10) appears as a (necessary) consistency condition, whereas in the CDT case it is satisfied as a non-trivial identity. Moreover, we have in the Euclidean string field theory setting the additional consistency test that $\int_0^\infty dt\mathcal{G}(l_1, l_2; t) = \mathcal{G}(l_1, l_2)$, where $\mathcal{G}(l_1, l_2)$ is the so-called universal loop-loop correlator calculated from matrix models [19, 20]. This was verified in [6].

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$Z(J)$ being the generating functional for universes disappearing into the vacuum. We now have

$$0 = -\lim_{t\to\infty} \frac{\partial}{\partial t} \langle 0|\mathrm{e}^{-t\hat{H}}\ \mathrm{e}^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle = \lim_{t\to\infty} \langle 0|\mathrm{e}^{-t\hat{H}}\ \hat{H}\ \mathrm{e}^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle. \tag{5.5}$$

Commuting the $\Psi(l)$'s in $\hat{H}$ past the source term effectively replaces these operators by $lJ(l)$, after which they can be moved to the left of any $\Psi^\dagger(l)$ and outside $\langle 0|$. After that the remaining $\Psi^\dagger(l)$'s in $\hat{H}$ can be replaced by $\delta/\delta J(l)$ and also moved outside $\langle 0|$, leaving us with a integro-differential operator acting on $Z(J)$,

$$0 = \int_0^\infty dl\, J(l)\, O\left(l, J, \frac{\delta}{\delta J}\right) Z(J), \tag{5.6}$$

where

$$\begin{aligned} O\left(l, J, \frac{\delta}{\delta J}\right) &= H_0(l)\frac{\delta}{\delta J(l)} - \delta(l) \\ &\quad -gl\int_0^l dl' \frac{\delta^2}{\delta J(l')\delta J(l-l')} - \alpha g l \int_0^\infty dl' l' J(l') \frac{\delta}{\delta J(l+l')}. \end{aligned} \tag{5.7}$$

The generating functional $Z(J,t)$ also includes totally disconnected universes which never "interact" with each other. Since our main interest is in universes whose space-time is connected, the appropriate generating functional $F(J,t)$ for connected universes is obtained by taking the logarithm of $Z(J,t)$, following standard quantum field-theoretic methods,

$$F(J,t) = \log Z(J,t). \tag{5.8}$$

From this we obtain the correlators

$$\langle 0|\mathrm{e}^{-t\hat{H}}\Psi^\dagger(l_1)\cdots\Psi^\dagger(l_n)|0\rangle_{\mathrm{con}} = \left.\frac{\delta^n F(J,t)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0}, \tag{5.9}$$

and it is straightforward to translate the Dyson-Schwinger equation (5.6)-(5.7) into an equation for the connected functional

$$F(J) = \lim_{t\to\infty} F(J,t), \tag{5.10}$$

namely,

$$\begin{aligned} 0 = \int_0^\infty dl\, J(l) \Bigg\{ & H_0(l)\,\frac{\delta F(J)}{\delta J(l)} - \delta(l) - gl\int_0^l dl'\, \frac{\delta^2 F(J)}{\delta J(l')\delta J(l-l')} \\ & -gl\int_0^l dl' \frac{\delta F(J)}{\delta J(l')}\frac{\delta F(J)}{\delta J(l-l')} - \alpha g l\int_0^\infty dl' l' J(l')\frac{\delta F(J)}{\delta J(l+l')} \Bigg\}. \end{aligned} \tag{5.11}$$

From eq. (5.11) one obtains the Dyson-Schwinger equation by differentiating (5.11) with respect to $J(l)$ a number of times and then taking $J(l) = 0$.

6. Application of the Dyson-Schwinger equation

Let us introduce the notation

$$w(l_1,\dots,l_n) \equiv \left.\frac{\delta^n F(J)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0}, \tag{6.1}$$

as well as the Laplace transform

$$w(x_1,\dots,x_n) \equiv \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \; \mathrm{e}^{-x_1 l_1 - \cdots - x_n l_n} w(l_1,\dots,l_n). \tag{6.2}$$

Next, differentiate eq. (5.11) with respect to $J(l)$ one, two and three times, then take $J(l)=0$ and Laplace-transform the resulting equations. This leads to the following three equations (recall that $H_0(x)f(x)=\partial_x[(x^2-\lambda)f(x)]$):

$$0 = H_0(x)w(x) - 1 + g\partial_x\Big(w(x,x)+w(x)w(x)\Big), \tag{6.3}$$

$$\begin{aligned} 0 = {} & (H_0(x)+H_0(y))w(x,y) + g\partial_x w(x,x,y) + g\partial_y w(x,y,y) \\ & +2g\left(\partial_x[w(x)w(x,y)]+\partial_y[w(y)w(x,y)]\right) + 2\alpha g\partial_x\partial_y\left(\frac{w(x)-w(y)}{x-y}\right), \end{aligned} \tag{6.4}$$

$$\begin{aligned} 0 = {} & (H_0(x)+H_0(y)+H_0(z))w(x,y,z) \\ & +g\partial_x w(x,x,y,z) + g\partial_y w(x,y,y,z) + g\partial_z w(x,y,z,z) \\ & +2g\partial_x[w(x)w(x,y,z)] + 2g\partial_y[w(y)w(x,y,z)] + 2g\partial_z[w(z)w(x,y,z)] \\ & +2g\partial_x[w(x,y)w(x,z)] + 2g\partial_y[w(x,y)w(y,z)] + 2g\partial_z[w(x,z)w(y,z)] \\ & +2\alpha g\left(\partial_x\partial_y\frac{w(x,z)-w(y,z)}{x-y} + \partial_x\partial_z\frac{w(x,y)-w(y,z)}{x-z} + \partial_y\partial_z\frac{w(x,y)-w(x,z)}{y-z}\right). \end{aligned} \tag{6.5}$$

The general structure of these equations should now be clear.[7] We can solve the Dyson-Schwinger equations iteratively. To this end, introduce an expansion of $w(x_1,\dots,x_n)$ in powers of the coupling constants g and α,

$$w(x_1,\dots,x_n) = \sum_{k=n-1}^{\infty} \alpha^k \sum_{m=k-1}^{\infty} g^m \; w(x_1,\dots,x_n;m,k). \tag{6.6}$$

The amplitude $w(x_1,\dots,x_n)$ starts with the power $(\alpha g)^{n-1}$ since we have to perform $(n-1)$ mergings during the time evolution in order to create a connected geometry if we begin with n separated spatial loops. One can find the lowest-order contribution to $w(x_1)$ from (6.3), use that to find the lowest-order contribution to $w(x_1,x_2)$ from (6.4), and then use this again in (6.5), which involves $w(x_1,x_2,x_3)$, etc. Returning to eq. (6.3), we can use the lowest-order expression for $w(x_1,x_2)$ to find the next-order correction to $w(x_1)$, use

[7] Interestingly, one can find a matrix model which reproduces the Dyson-Schwinger equations, [21] This indeed open up the possibilities of using the standard loop equations of matrix models, either in the original form for higher genus surfaces given in [27] or in the more modern version found in [28 – 31]. This is discussed in detail in [21] where one can also find a discussion of the formal link to topological string theory and the Dijkgraaf-Vafa model.

this and the lowest-order correction for $w(x_1, x_2, x_3)$ to find the next-order correction to $w(x_1, x_2)$, etc.

Two remarks are in order: first, the integration constants arising during the integration of (6.3)-(6.5) and the corresponding higher-order equations are uniquely fixed by the requirement that the correlation functions fall off as the lengths $l_i \to \infty$, i.e. the requirement that the Laplace-transformed amplitude $w(x_1, \ldots, x_n)$ be analytic for $x_i > 0$. Second, the expressions obtained for $w(x_1, \ldots, x_n)$ can of course be obtained directly from a diagrammatic expansion, using the interaction rules shown in figure 3, where the propagation is defined by $\hat{H}_0$, and then integrating in a suitable way over the times t_i involved. The results for the first few orders are

$$w(x;0,0) = \frac{1}{x+\sqrt{\lambda}}, \tag{6.7}$$

$$w(x;1,0) = \frac{x+3\sqrt{\lambda}}{4\lambda(x+\sqrt{\lambda})^3}, \tag{6.8}$$

$$w(x,y;1,1) = \frac{1}{2\sqrt{\lambda}(x+\sqrt{\lambda})^2(y+\sqrt{\lambda})^2}, \tag{6.9}$$

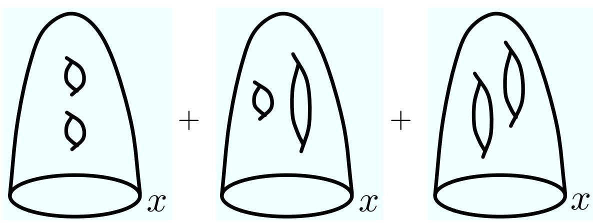
$$w(x,y,z;2,2) = \frac{7\lambda^{\frac{3}{2}}+5\lambda(x+y+z)+3\sqrt{\lambda}(xy+xz+yz)+xyz}{4\lambda^{\frac{3}{2}}(\sqrt{\lambda}+x)^3(\sqrt{\lambda}+y)^3(\sqrt{\lambda}+z)^3}. \tag{6.10}$$

For all of these amplitudes, the space-time topology is trivial. To lowest order in g, i.e. without any additional baby universes, and using the results (6.7)-(6.10) in the iteration as described above, the genus-one and genus-two amplitudes become

$$w(x;2,1) = \frac{15\lambda^{\frac{3}{2}}+11\lambda x+5\sqrt{\lambda}x^2+x^3}{32\lambda^{\frac{5}{2}}(\sqrt{\lambda}+x)^5}, \tag{6.11}$$

$$\begin{aligned} w(x;3,2) = \frac{1}{2048\lambda^{\frac{11}{2}}(\sqrt{\lambda}+x)^9}\Big(&11319\lambda^{\frac{7}{2}}+19951\lambda^3 x+21555\lambda^{\frac{5}{2}}x^2+ \\ &16955\lambda^2x^3+9765\lambda^{\frac{3}{2}}x^4+3885\lambda x^5+945\sqrt{\lambda}x^6+105x^7\Big). \end{aligned} \tag{6.12}$$

In a diagrammatic notation, the genus-two amplitude $w(x;3,2)$ corresponds to the following three diagrams (including suitable integrations over the times t_i):

$w(x;3,2) =$  .

As mentioned above, the expansion of the amplitude $w(x_1, \ldots, x_n)$ starts with the power $(\alpha g)^{n-1}$, coming from merging the n disconnected spatial universes. The remaining powers of α are associated with a non-trivial space-time topology in the form of h additional "handles" on the connected world sheet. From a purely Euclidean point of view this suggest a reorganization of the series according to

$$w(x_1,\ldots,x_n) = (\alpha g)^{n-1}\sum_{h=0}^{\infty}(\alpha g^2)^h w_h(x_1,\ldots,x_n), \tag{6.13}$$

$$w_h(x_1,\ldots,x_n) = \sum_{j=0}^{\infty} g^j w(x_1,\ldots,x_n; n-1+2h+j, n-1+h), \tag{6.14}$$

amounting to a topological expansion in αg^2, solving at each order for all possible baby-universe creations which at some point will vanish into the vacuum. This implies that $w_h(x_1,\ldots,x_n)$ will be a function of g, although we do not write the dependence explicitly.

The Dyson-Schwinger equations allow us to obtain the topological expansion iteratively, in much the same way as in our earlier power expansion in g. Since we have $w(x,x) = O(\alpha)$, this term does not contribute to lowest order; from eq. (6.3) we obtain a closed equation for $w_0(x)$, namely,

$$H_0(x)w_0(x) + g\partial_x w_0^2(x) = 1. \tag{6.15}$$

This equation is of course just eq. (4.4), where we have made the identification

$$w_0(x) = W_{g,\lambda}(x). \tag{6.16}$$

Knowing $w_0(x)$ allows us to obtain $w_0(x,y)$ from (6.4), since $w(x,y,z)$ is of order $O(\alpha^2)$. Therefore the three-loop term does not contribute to the lowest order in α of eq. (6.4), which is $O(\alpha)$, and we find that to lowest order

$$\Big(H_0(x)+2g\partial_x w_0(x)+H_0(y)+2g\partial_y w_0(y)\Big)w_0(x,y) = -2\partial_x\partial_y\left(\frac{w_0(x)-w_0(y)}{x-y}\right). \tag{6.17}$$

We conclude that $w_0(x,y)$ is entirely determined by the knowledge of $w_0(x)$. Note that using the definition (2.10) we can simplify (6.17) to

$$\frac{\partial}{\partial x}\Big(\hat{W}_{\lambda,g}(x)w_0(x,y)\Big) + \frac{\partial}{\partial y}\Big(\hat{W}_{\lambda,g}(y)w_0(x,y)\Big) = -\frac{1}{g}\frac{\partial^2}{\partial x\partial y}\left(\frac{\hat{W}_{\lambda,g}(x)-\hat{W}_{\lambda,g}(y)}{x-y}\right). \tag{6.18}$$

The solution $w_0(x,y)$ can readily be found from eq. (6.18), yielding

$$w_0(x,y) = \frac{1}{f(x)f(y)}\frac{1}{4g}\left(\frac{[(x+c)+(y+c)]^2}{[f(x)+f(y)]^2} - 1\right), \tag{6.19}$$

where

$$f(x) = \sqrt{(x+c)^2 - 2g/c} = \hat{W}_{\lambda,g}(x)/(x-c). \tag{6.20}$$

In fact, this solution was already found in [14] since we have by definition that

$$w_0(x,y) = \int_0^\infty dt\; \mathcal{G}_{\lambda,g}(x,y;t), \tag{6.21}$$

where $\mathcal{G}_{\lambda,g}(x,y;t)$ is the Laplace transform of $\mathcal{G}(l_1,l_2;t)$ defined in (4.11), with $t_1 = t/2$. This function is precisely the *loop-loop function* of [14]. When expanded to lowest order in g, it reproduces (6.9).

As should by now be clear, one can iterate the Dyson-Schwinger equations in a systematic way as a power series in the number h of handles of the world sheet, exactly like we iterated them as a function of the coupling constant g, leading to

$$\begin{aligned} w(x) &= w_0(x) + \alpha g^2 w_1(x) + \alpha^2 g^4 w_2(x) + \cdots, \\ w(x,y) &= \alpha g w_0(x,y) + \alpha^2 g^3 w_1(x,y) + \cdots, \end{aligned} \tag{6.22}$$

etc. As an instructive example we will calculate the genus-one amplitude $w_1(x)$. While eq. (6.15) was the 0th order in α of eq. (6.3), the 1st order reads

$$\frac{\partial}{\partial x}\left(\hat{W}_{\lambda,g}(x) w_1(x) + w_0(x,x)\right) = 0, \tag{6.23}$$

where $w_0(x,x)$ is given by eq. (6.19). The integration constant is fixed by the requirement that $w_1(x)$ be analytic for $x > 0$, i.e. that $w_1(l)$ fall off as $l \to \infty$. We obtain

$$w_1(x) = \frac{w_0(c,c) - w_0(x,x)}{\hat{W}_{\lambda,g}(x)} = \frac{(x+3c)(x^2+2cx+5c^2-4g/c)}{2c(4c^2-2g/c)^2((x+c)^2-2g/c)^{5/2}}, \tag{6.24}$$

which upon expansion in powers of g to lowest order reproduces (6.11), as one would expect.

7. Discussion

In the present work, we have developed a string field theory in zero-dimensional target space, based on the CDT quantization of two-dimensional quantum gravity. It shares many properties of the non-critical string field theory originally defined in [6], from which we borrowed the formalism in the first place. Yet, our results are different and in some ways simpler. The tadpole term in our case is simply $\rho(l) = \delta(l)$, encoding the fact that universes can only disappear into the vacuum if they have zero spatial volume (that is, zero length). This is in accordance with the interaction between spatial universes, which also preserves the total length. In non-critical string field theory the evolution in proper time results in a process where the original spatial universe at proper time $t = 0$ spawns an infinity of (infinitesimal) baby universes during the time evolution. This is related to the fact that the proper time in non-critical string field theory has the anomalous length dimension 1/2. In our new CDT-based string field theory the situation is different. The proper time t has canonical dimension 1, and the number of baby universes created during the time evolution is finite [14].

It is not possible to connect the non-critical string field theory and the CDT-based string field theory by a simple analytic continuation in the coupling constant g, not even in the limit as $\alpha = 0$ [14]. It was demonstrated in [9] that, starting from a discretized, regularized version of the theory, the Euclidean theory (quantum Liouville theory) is obtained if the "bare" dimensionless coupling constant g_0 is of order one. However, the relation between the bare coupling constant and the dimensionful continuum coupling constant g used in the present article is given by

$$g_0 = g a^3, \tag{7.1}$$

where a is the lattice spacing in the dynamical triangulations providing the regularization (c.f. section 2). As discussed in [14], the generalized CDT continuum limit corresponds to g fixed, $a \to 0$, and thus to $g_0(a) \to 0$. The fact that $g_0(a)$ goes to zero in the CDT string field theory is of course related to the finite number of baby universes generated in this theory. By contrast, we have an infinite number of baby universes generated in non-critical string field theory, where g_0 is of order one.

However, there is clearly a deeper connection between the Euclidean and the CDT theory awaiting to be fully understood. It was shown in [16] that by integrating out the "excessive outgrowth" of baby universes in Euclidean 2d quantum gravity, one recovers the CDT theory, and the mapping between the dimensionless variables $x/\sqrt{\lambda}$ of the two theories was given explicitly. This mapping was later discovered by Seiberg and Shih [22] as the uniformization map from the algebraic surface representing the "semiclassical" non-critical string to the complex plane. The singular points of this algebraic surface correspond to so-called ZZ-branes, where there is a transition from compact to non-compact topology [23]. These singular points are mapped to points in the complex plane where one has a similar transition from compact to non-compact geometry in the CDT context [24].

It would be interesting to generalize the present string field theory based on causal dynamical triangulations to include the coupling to matter. In particular, one would like to investigate whether this theory still exhibits any trace of the presence of a $c = 1$ barrier. Since the existence of this barrier in the Euclidean theory can be partly understood as the result of an excessive creation of baby universes, tearing apart the two-dimensional worldsheet [5, 25], it is clear that the CDT theory may behave differently. Numerical simulations are compatible with the presence of a barrier at large values of the conformal charge c [26], but no definite results are available at this stage. Work is in progress on determining whether the CDT string field theory can provide a useful analytic tool in addressing this situation.

Equally interesting is the possibility of performing a summation over world sheets of all genera. Again, since the double-scaling limit in CDT string field theory is different from the double-scaling limit in non-critical string theory, and since there is a larger "penalty" for creating a higher-genus surface in the sense outlined above, viewing the creation of a higher-genus world sheet as a successive creation and annihilation of a baby universe, one could hope that the result of such a summation was better behaved and less ambiguous than was the case in non-critical string theory. Work in this direction is also in progress.

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